J. Fluid Mech. (1971), *vol.* 46, *part* 2, *pp.* 385–394
Printed in Great Britain

Zero and negative entrainment in turbulent shear flow

By M. R. HEAD
Indian Institute of Technology, Kanpur†

AND P. BRADSHAW
Department of Aeronautics, Imperial College, London

(Received 8 April 1970 and in revised form 21 July 1970)

In certain accelerated flows the entrainment in the boundary layer, as normally defined, may be either zero or negative; on the other hand, there is no reason to suppose, on physical grounds, that the spread of mean or fluctuating vorticity should cease or become negative in such flows. This paradox is resolved in the present paper. It is also shown that in the equilibrium turbulent sink-flow boundary layer, where the entrainment as normally defined is zero, the reduced advection along streamlines in the outer part of the layer comes about mainly through increased dissipation: there is no reason to assume any radical change in the turbulence structure.

1. Introduction

If we consider a turbulent shear flow, such as a turbulent jet or boundary layer, it seems reasonable to suppose that the turbulence should spread with time (i.e. with distance downstream) into the surrounding fluid, and that mean as well as fluctuating vorticity, should be transferred to the surrounding fluid in this way. Even if the turbulence decays with distance downstream, we should expect the propagation of mean vorticity to proceed by the normal action of viscosity.

From this point of view it would appear that the entrainment should always be positive, i.e. that the mass flow 'infected' with mean vorticity should continually increase with distance downstream, in either laminar or turbulent flow.

Nevertheless, there are flows where this is apparently not the case. In particular, it is well known that if a turbulent boundary layer is subjected to a sufficiently strong and sustained favourable pressure gradient it will revert to the laminar form and in the process the mass flow in the boundary layer may actually decrease. Whilst it is not difficult to accept the decay of turbulence in this particular situation, it is less easy to see how the amount of fluid infected with mean vorticity (and hence possessing velocity defect) can decrease, since vorticity of appropriate sign is continually being generated at the wall and diffused outward. Again, in the special case of the laminar or turbulent equilibrium boundary layer on the plane walls of a converging channel with a line sink at the apex (sink flow: see inset to figure 1), it appears inescapable that the entrainment should be zero. From continuity the Reynolds number of the flow is the same at all points

† Permanent address: Engineering Department, University of Cambridge.

along the channel and similarity demands that all streamlines should be radial lines; thus the proportion of the total mass flow contained within the boundary layer should remain constant. We therefore have the situation where a shear flow is apparently neither decaying nor encroaching upon the external irrotational flow.

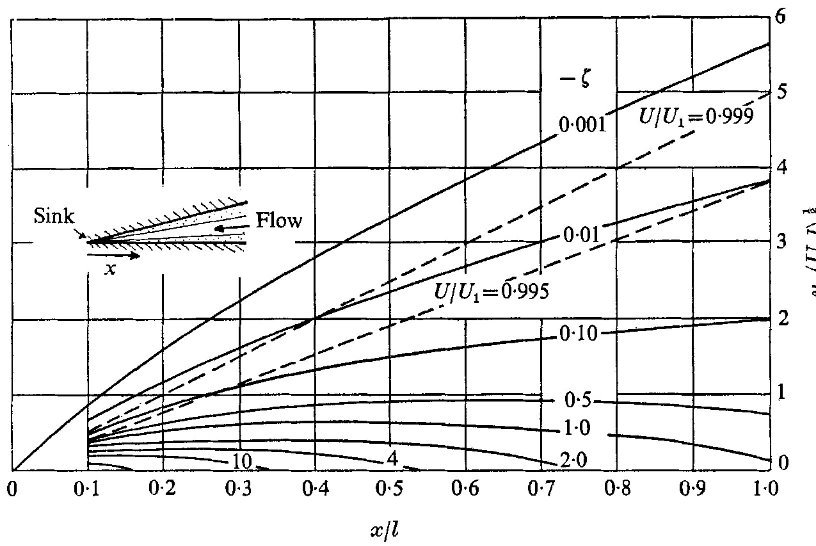


FIGURE 1. Approximate contours of constant vorticity in two-dimensional laminar sink flow: flow from right to left. $-\zeta = (\partial U/\partial y)\,(l/U_0)\,(\nu/U_0 l)^{\frac{1}{2}}$; U_0 = reference velocity; l = reference length.

In what follows we show that, in fact, such encroachment does occur, and that, although the rate at which it proceeds is small, the eddy structure in the case of a turbulent boundary layer in sink flow does not differ radically from that in a normal turbulent boundary layer. We also show that in the case of turbulent boundary layers reverting to the laminar form there is no basic conflict between the negative entrainment that would normally be said to occur and the continuous increase in the amount of fluid infected with mean vorticity that physical considerations would appear to demand.

2. The equilibrium boundary layer in sink flow

2.1. *General considerations*

It is instructive to consider first the solution for laminar boundary-layer flow in this situation, given in Rosenhead (1963, page 236). Figure 1 shows contours of constant ζ, the vorticity made dimensionless by a constant (arbitrary) reference velocity U_0, and a constant reference length l defined as the distance from the sink at which $U_1 = U_0$. It will be seen that these contours cross the streamlines (which are also lines of constant U/U_1) at an appreciable angle: therefore the mass flow infected with mean vorticity greater than some reference value, $-\zeta = 0{\cdot}01$, say, continually increases with distance downstream, in agreement with the intuitive ideas of positive entrainment expressed at the beginning of § 1.

However, if the vorticity is made dimensionless with respect to *local* scales (say the local free-stream velocity U_1 and the boundary-layer thickness δ) then the corresponding contours of $-(\delta/U_1)\,\partial U/\partial y$ coincide with the streamlines, and the mass flow infected with a value of $-(\delta/U_1)\,\partial U/\partial y$ greater than some reference value remains constant, in agreement with the requirements of self-preservation.

It will be seen that there is no conflict between the continually increasing proportion of the flow infected with mean vorticity greater than some constant reference value and the constant proportion of the flow that comprises the boundary layer as normally defined. Similarly, it is to be expected that, in the equilibrium turbulent boundary layer in sink flow, the fluctuating vorticity will affect an ever-increasing proportion of the flow, although the mass flow confined within a particular value of $-(\delta/U_1)\,\partial U/\partial y$ or of $(\delta/U_1)\sqrt{(\text{mean square vorticity})}$ will remain constant.

2.2. *Turbulent energy balance*

Although the foregoing discussion removes a major difficulty in showing that nominally zero entrainment is not incompatible with the outward spread of mean and fluctuating vorticity, it remains true that the rate at which fluctuating vorticity is propagated into the surrounding fluid in the turbulent sink flow is very much less than in a more typical turbulent boundary layer, although the turbulence level over most of the layer is not very different. This prompts the question whether the diffusion of turbulent energy towards the edge of the layer is qualitatively different in highly accelerated flows.

Bradshaw, Ferriss & Atwell (1967) presented a method of calculating turbulent boundary development by the use of the turbulent energy equation,

$$\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\tfrac{1}{2}\overline{q^2}-\frac{\tau}{\rho}\frac{\partial U}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\overline{p'v}}{\rho}+\tfrac{1}{2}\overline{q^2v}\right)+\epsilon=0,$$

where the terms represent respectively advection, $(-1)\times$production, loss by diffusion and dissipation; here $q^2 = u^2+v^2+w^2$. The method uses certain assumptions about the universality of the turbulence structure, specifically about relations between the shear stress τ and other properties of the turbulence, and it is therefore interesting to see whether it can deal satisfactorily with the present flow. The results of a calculation are shown in figure 2: the Reynolds number should be constant and the slight increase occurs solely because the initial velocity and shear stress profiles were taken from an experiment (Herring & Norbury 1967) where it increased very significantly. The calculated profiles, which are not shown here, agree well with the results of Launder & Stinchcombe (1967) at the highest Reynolds number of their experiments. The assumption of universal relations for the turbulence structure is evidently satisfactory and we do not have to postulate any unusual behaviour of the turbulence to explain the very low rate at which fluctuating vorticity is propagated outwards in this case.

Figure 3(*a*) shows the calculated turbulent energy balance for the same boundary layer, near the outer edge of the flow: the results show up some finite-difference errors in the computer program but are adequate for the present purpose. Figures 3 (*b*) and 3 (*c*) show the calculated energy balances for a boundary layer in zero pressure gradient and a strongly retarded self-preserving boundary layer respectively: the experimental energy balances for these cases are given by Bradshaw (1967) and agree qualitatively with those shown here, although the experimental results are incomplete and not very accurate.

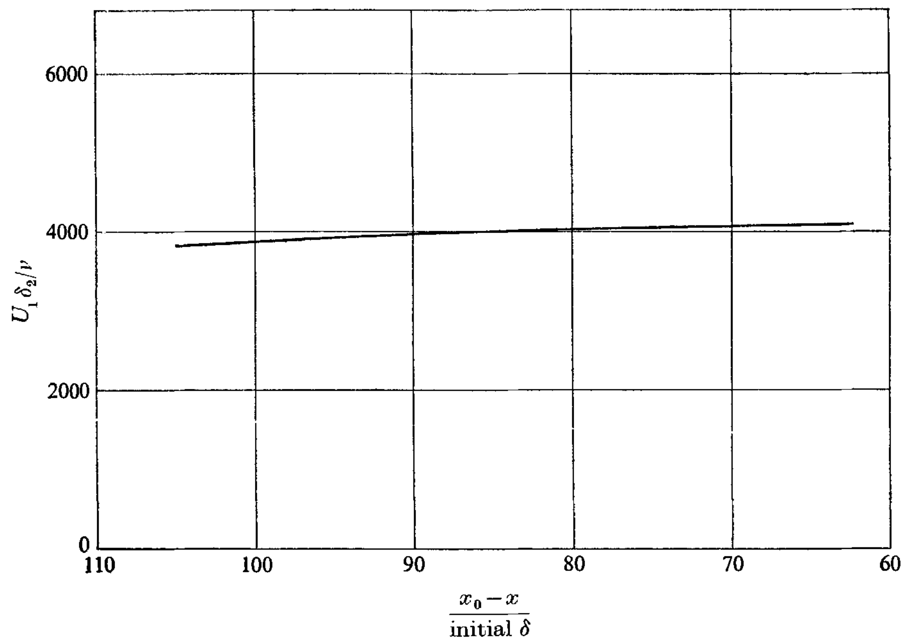


FIGURE 2. Near-constancy of calculated Reynolds number in turbulent sink flow. x positive downstream, $x = x_0$ at apex, $U_1 \propto 1/(x_0 - x)$. Calculation started at $(x_0 - x)/(\text{initial } \delta) = 105$.

In figures 3 (*b*) and 3 (*c*), and in other boundary layers where the entrainment, however defined, is unequivocally positive, the diffusion of turbulent energy towards the outer edge supplies the advection or rate of increase of turbulent energy along a streamline (which may normally be taken as a measure of the entrainment) and both production and dissipation are negligible near the outer edge. However, we see from figure 3 (*a*) that near the edge of the sink flow advection is very much smaller than the dissipation, so that the latter absorbs nearly all the diffusion. The behaviour of the diffusion itself does not appear to be qualitatively different from that in an ordinary flow.

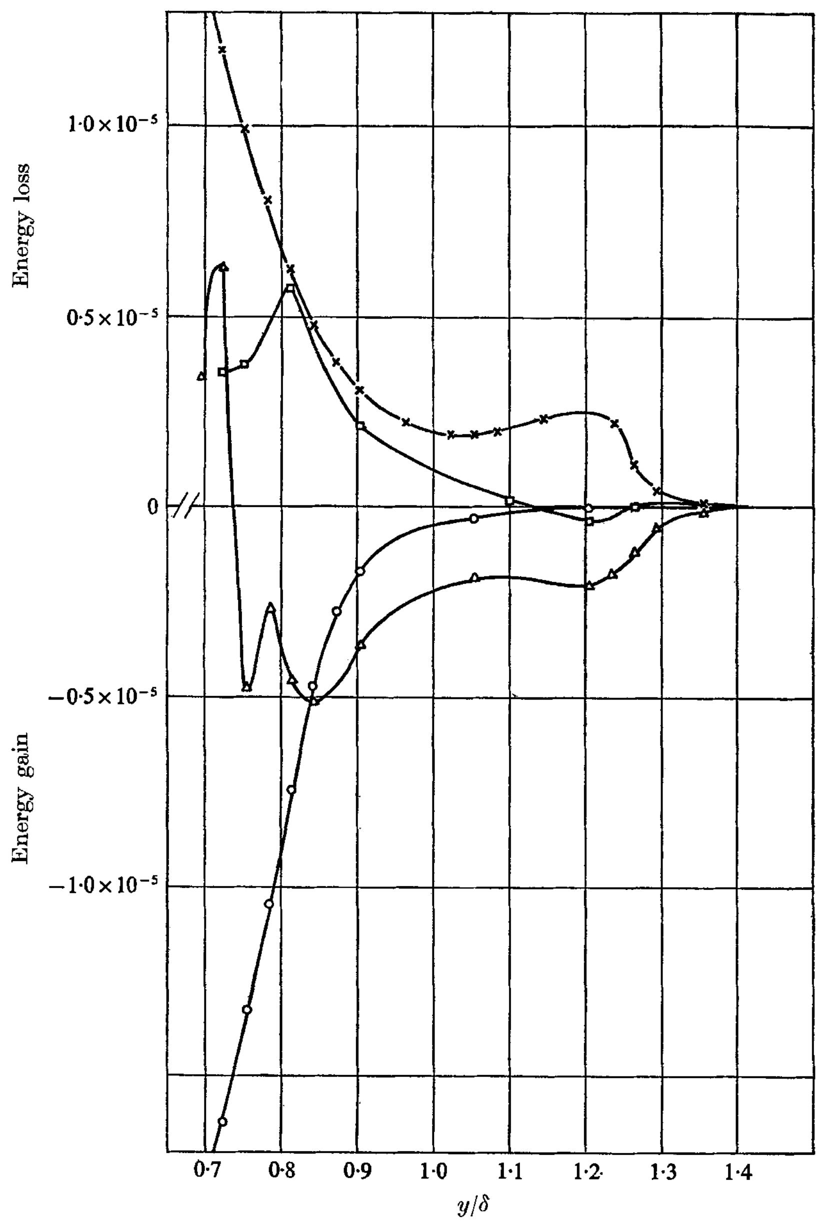
The values of (advection)/(dissipation) shown in figure 3 depend on the details of the calculation method but the qualitative result may be confirmed by inserting plausible magnitudes in a simple analysis. In the sink flow,

$$U = U_1 f(y/(x - x_0))$$

and the turbulent energy $\frac{1}{2}\rho\overline{q^2}$ is $\rho U_1^2 g(y/(x - x_0))$ with $x_0 > x$. The same expressions, with different f and g and with $x_0 < x$, are an adequate approximation for the present purpose in a boundary layer in zero pressure gradient if x_0 is chosen to match the local rate of growth of boundary-layer thickness δ. In both flows $\delta/(x-x_0)$ is numerically of order 0·01.

The advection is

$$\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\tfrac{1}{2}\overline{q^2}.$$



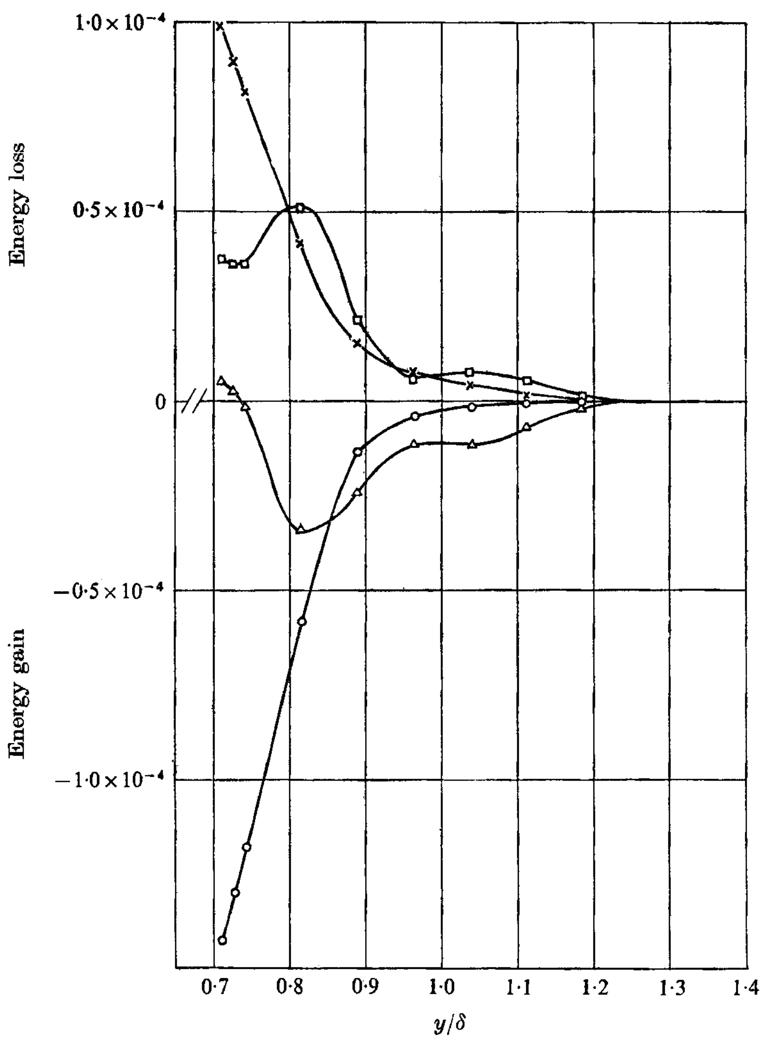
(*a*)

FIGURE 3. For legend see p. 391.

Since $\partial\overline{q^2}/\partial x = -(y/(x-x_0))\,\partial\overline{q^2}/\partial y$ if U_1 = constant, the advection near the edge of the boundary layer in zero pressure gradient, where $U \doteqdot U_1$ and $V \doteqdot V_1$, becomes

$$\left(V_1 - \frac{y}{x-x_0}U_1\right)\frac{\partial}{\partial y}(\tfrac{1}{2}\overline{q^2}) \text{ which is of order } -0{\cdot}01U_1\frac{\partial}{\partial y}(\tfrac{1}{2}\overline{q^2}).$$

Now near the edge of any boundary layer, $\overline{q^2}$ decreases very rapidly with y, $\overline{q^2} \propto y^{-n}$ say, where n is very large: also the turbulent energy in the rotational part of the flow is a roughly constant multiple, $\frac{1}{2}a_1$ say, of the shear stress, a_1 being roughly 0·15. Therefore the advection is of order $0{\cdot}005n\tau U_1/\rho a_1\delta$ near $y = \delta$. According to the assumptions in the calculation method of Bradshaw *et al.* (1967) the dissipation near $y = \delta$ is about $(\tau/\rho)^{\frac{3}{2}}/0{\cdot}04\delta$ and, accepting this



(*b*)

FIGURE 3. For legend see facing page.

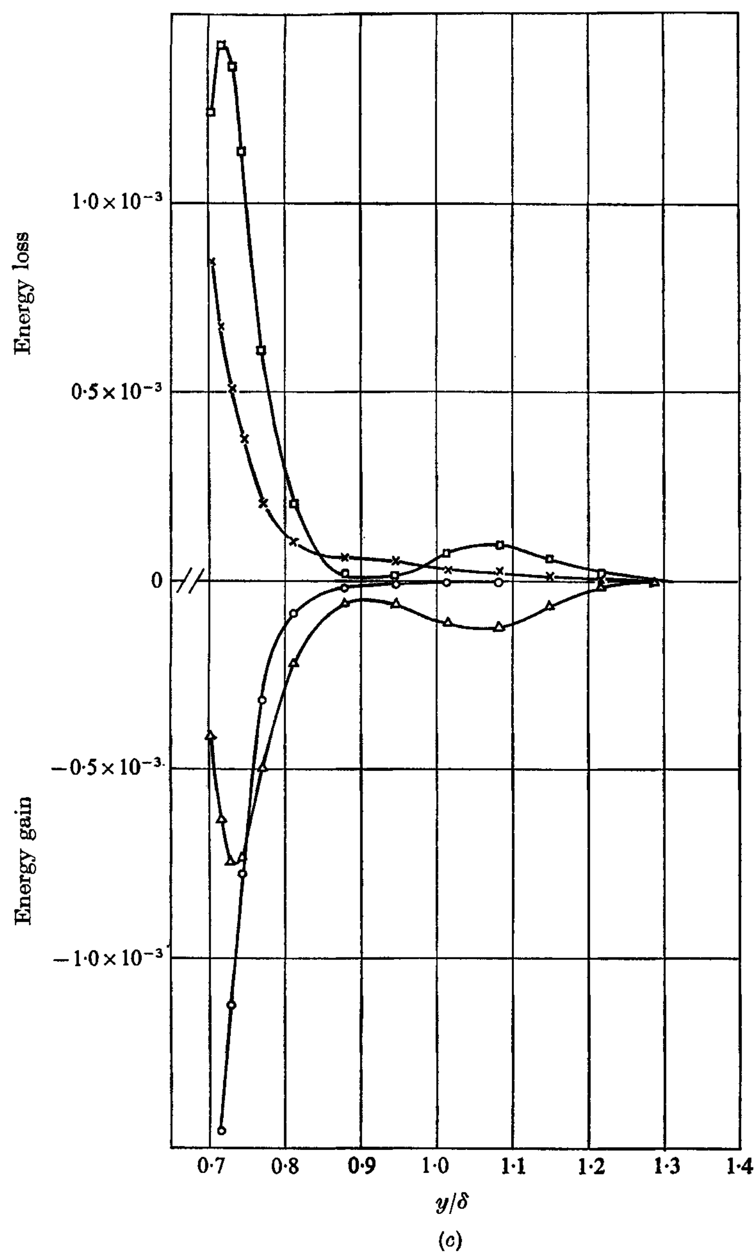


FIGURE 3. Calculated energy balance near edge of boundary layer: (*a*) sink flow between converging planes; (*b*) zero pressure gradient; (*c*) strongly retarded flow $U_1 \propto x^{-0.255}$. □, advection; ×, dissipation; △, diffusion; ○, production. Quantities made dimensionless with U_1 and δ.

value for the purposes of the present order of magnitude argument, we get

$$\frac{\text{advection}}{\text{dissipation}} \sim 2\times 10^{-4}\frac{U_1}{a_1(\tau/\rho)^{\frac{1}{2}}}n,$$

near the edge of a boundary layer in zero pressure gradient. In the sink flow $\overline{q^2}/U_1^2$ is constant along a given streamline so that $\partial\overline{q^2}/\partial y$ does not appear in the advection and, assuming that the same formula for dissipation applies in the postulated absence of large changes of turbulence structure, we get

$$\frac{\text{advection}}{\text{dissipation}} \sim 4\times 10^{-4}\frac{U_1}{a_1(\tau/\rho)^{\frac{1}{2}}},$$

which is much smaller, since $n \gg 1$ and $\tau/\rho U_1^2$ is larger than in zero pressure gradient in the outermost part of the boundary layer (see figure 4).

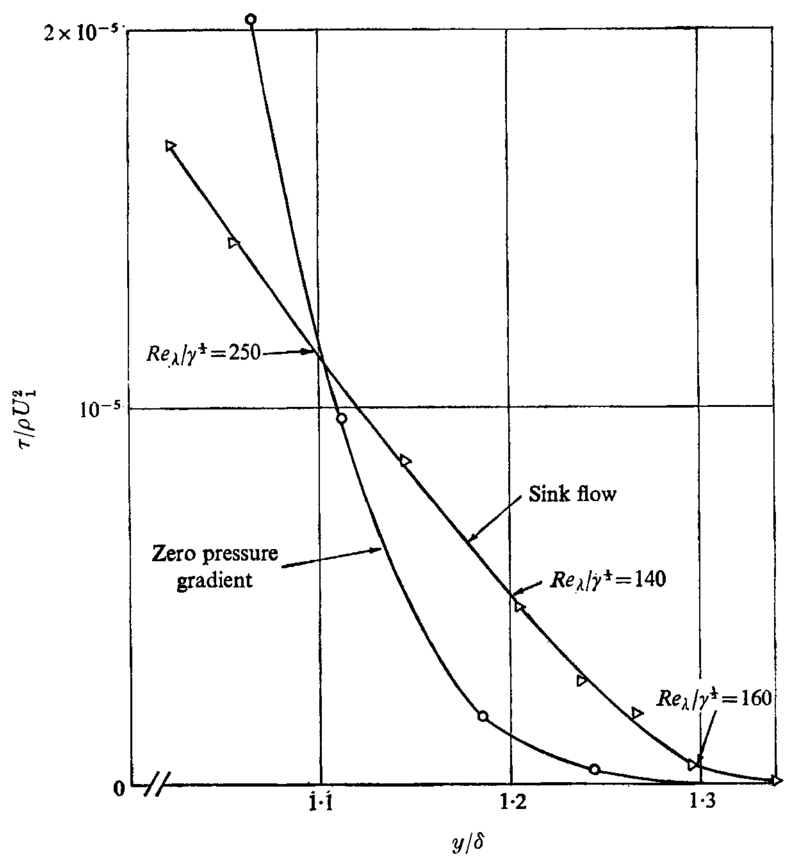


FIGURE 4. Calculated shear stress profiles near edge of boundary layer.

The calculations and the analysis therefore show that, in the case of sink flow, the reduction of advection comes about through the increase of dissipation rather than the reduction of diffusion, and no radical change in turbulence structure is required to account for this behaviour. In particular there would seem to be no justification for the hypothesis that large eddies are absent from this flow, although they are undoubtedly weaker than in a boundary layer in zero pressure gradient, as evidenced by the low level of shear stress over most of the layer.

3. More highly accelerated flows

Applying here the same general considerations as were applied in the case of sink flow, we may again reasonably assume that both mean and fluctuating vorticity are continually being propagated into the surrounding fluid, so that, on the absolute level, an increasing quantity of the flow is being affected, and the entrainment in this sense remains positive.

On the relative level, however, where the magnitudes of mean and fluctuating vorticity are made non-dimensional by the use of local quantities, we may now expect that, instead of remaining constant as in the case of sink flow, the quantity flow beneath contours of constant vorticity made non-dimensional in this way will actually decrease in the streamwise direction. Thus the quantity flow in the boundary layer, as normally defined, will have decreased and we may reasonably refer to negative entrainment in this situation, while recognizing that this comes about not through any reduction in the quantity of fluid infected with mean or fluctuating vorticity, but through our ceasing to regard as part of the boundary layer any fluid in which the level of these quantities has fallen sufficiently low.

We are justified in assuming the decay of turbulence, in relative terms, in turbulent flows that are more highly accelerated than the sink-flow boundary layer, since in the latter case the turbulence is only just self-sustaining (again in relative terms: the actual turbulent intensity along streamlines must in fact increase as U_1^2). Whether the decay proceeds sufficiently far for the boundary layer to revert effectively to the laminar condition will depend upon the initial boundary-layer Reynolds number, the strength of the favourable pressure gradient and the distance over which it is applied.

The decay of turbulence will still be described by the turbulent energy equation, but, except in the initial stages of the decay process, we should not be justified in assuming the universal relations used in the calculation method that was satisfactorily applied to the sink-flow boundary layer: also, viscous terms neglected in the above form of the equation will become important. There is no conflict between the decay of turbulence and its continuing spread into the surrounding fluid; contours of constant turbulence intensity expressed in relative terms will certainly contract and in absolute terms *may* do so, but even with the effective disappearance of small scale turbulence the larger scales will continue to infect the surrounding fluid until they too effectively disappear, leaving behind only a mean velocity defect that may be quite insignificant for practical purposes.

An overall picture of the changes occurring in the highly accelerated layer may be obtained by making rather gross simplifying assumptions. Let us first assume that the accelerations are sufficiently rapid for Bernoulli's equation to be applied along streamlines in the outer part of the layer. From this it can be simply shown that the non-dimensional velocity defect $\Delta U/U_1$ should decrease approximately as $1/U_1^2$. If we further assume that the absolute turbulent intensity $\overline{q^2}$ remains constant along streamlines,† then it follows that $\overline{q^2}/U_1^2$ should also decrease as $1/U_1^2$. It is therefore not surprising that, in wind-tunnel contractions and the like, where U_1 is increased by a factor of 5 or 10, initially quite large velocity defects and high turbulence levels should be rendered effectively negligible compared with the vorticity which has been newly created and diffused out to only a short distance from the wall.

4. Conclusions

To resolve conceptual diffculties associated with the terms 'zero' and 'negative' entrainment it is only necessary to observe the distinction between the use of the word entrainment to mean the spread of velocity defect or mean vorticity into the surrounding fluid and its use as representing the rate of increase of mass flow in the 'boundary layer' as normally defined: used in the first sense, the entrainment can never be other than positive, though it may be very small; used in the second sense it may well be negative in highly accelerated flows. The presence or absence of turbulence, or its progressive decay, would not appear to affect these basic considerations.

From the fact that a calculation method which makes use of the turbulent energy equation and relationships derived from normal turbulent boundary layers has been successfully applied to the calculation of the sink-flow boundary layer, it is concluded that the turbulence structure of this layer is not qualitatively different from that of a normal boundary layer; the reduced advection is seen to result from an increase in dissipation rather than a reduction in diffusion. The utility of the calculation method in more highly accelerated flows is probably restricted to the initial stages of the decay of turbulence.

REFERENCES

Bradshaw, P. 1967 *J. Fluid Mech.* **29**, 625.

Bradshaw, P., Ferriss, D. H. & Atwell, N. P. 1967 *J. Fluid Mech.* **28**, 583.

Herring, H. J. & Norbury, J. F. 1967 *J. Fluid Mech.* **27**, 541.

Launder, B. E. & Stinchcombe, H. S. 1967 *Imperial College* TWF/TN/21.

Rosenhead, L. 1963 *Laminar Boundary Layers*. Cambridge University Press.

† Recent data of Kovasznay & Blackwelder (private communication) show that this is a good approximation in highly accelerated flows.